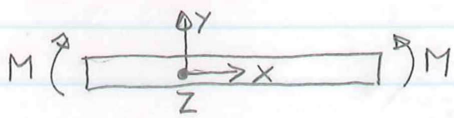
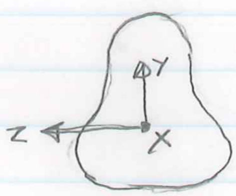
15. Pure bending with a symmetric cross-section

This section deals with bending under constant bending moment. There is no resultant shear or axial force on a cross-section and no resultant torque. The material is linearly elastic, homogeneous and isotropic.

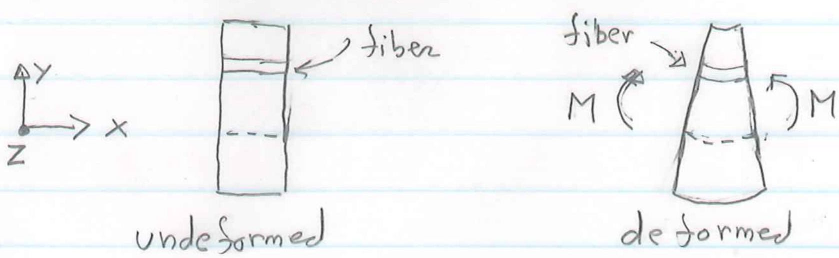


x is the axis along the beam.
y-z is the cross-sectional plane.
The bending moment M is shown positive.



This shows a cross-section. Symmetry exists about the x-y plane. The bending moment is about the z axis.

Examine a length of the beam.



The cross-sections rotate and remain plane.

Under positive M, the fibers in the top portion of the beam get shorter and those in the bottom portion get longer. Because the cross-sections remain plane, ε_x is a linear function of y. Along the dashed line, ε_x is zero. This line is called the neutral axis and is where the z axis is located. (The x-z plane is the neutral plane.) For positive M, ε_x is compressive for positive y and tensile for negative y.

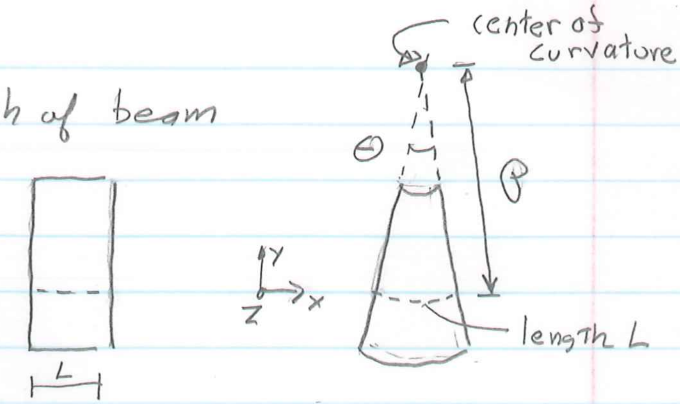
Every fiber in a beam under constant bending moment deforms into the arc of a circle. Let ρ be the radius of curvature of the fibers in the deformed x-z plane, where

$\varepsilon_x = 0$. Also, L = length of beam and θ = arc angle.
We have

$$\varepsilon_x = \frac{L - (\rho - y)\theta}{L}$$

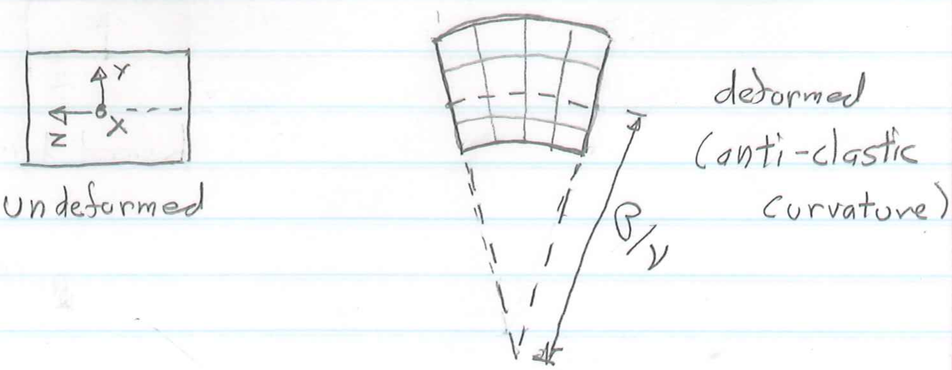
$$= -\frac{y}{\rho}$$

since $\theta = \frac{L}{\rho}$.



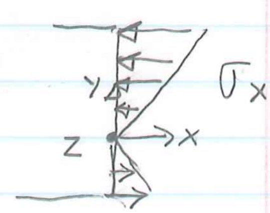
undeformed deformed

It can be shown that pure bending produces only σ_x; the other stress components are all zero. In addition to ε_x, the other strains are $\varepsilon_y = \varepsilon_z = -\nu\, \varepsilon_x$ and zero for the shear strains. Because of the presence of ε_y and ε_z, the cross-section changes shape during bending. For example, consider a rectangular cross-section.



undeformed deformed (anti-clastic curvature)

Over the cross-section, $\sigma_x = E\, \varepsilon_x = -E \frac{y}{\rho}$. For positive moment, compressive stresses (and strains) occur above the neutral axis and tensile stresses (and strains) occur below. The variation is linear with y.



Since the resultant axial force (x direction) on a cross-section is zero:

$$\int_{c-s} \sigma_x \, dA = \frac{-E}{\rho} \int_{c-s} y \, dA = 0 ,$$

which implies the neutral axis (z axis) is a centroidal axis for a cross-section.

We would like to relate the bending moment M to the stresses σ_x.

$$M = \int_{c-s} -y\,\sigma_x\,dA = \frac{E}{\rho}\int_{c-s} y^2\,dA = \frac{EI}{\rho}$$

where $I = \int_{c-s} y^2\,dA$ is the moment of inertia of the cross-section. (y measured from the neutral axis) In the above equation, ρ is replaced by

$$\rho = -\frac{y}{\varepsilon_x} = -\frac{Ey}{\sigma_x} \quad \text{to give}$$

$$\sigma_x = -\frac{My}{I}$$

Define $c = |y_{max}|$, the distance from the neutral axis to the top or bottom fiber, whichever is greater.
Then

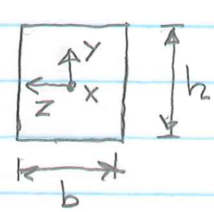
$$\sigma_{max} = \frac{Mc}{I},$$ the maximum normal stress, whether tension or compression.

This last equation is sometimes written as

$$\sigma_{max} = \frac{M}{S}$$ where S is the section modulus $= \frac{I}{c}$.

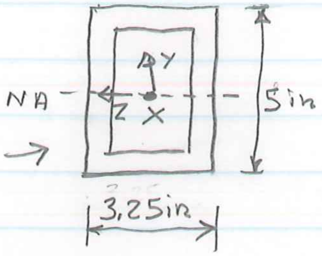
For example, for a rectangular cross-section,

$$S = \frac{\frac{1}{12}bh^3}{\frac{h}{2}} = \frac{1}{6}bh^2$$



Example:

cross-section →



wall thickness = 0.25 in

$E = 10.6 \times 10^6$ psi

$c = 2.5$ in

$I = \frac{1}{12}(5\,in)^3 \cdot 3.25\,in - \frac{1}{12}(4.5\,in)^3 \cdot 2.75\,in$

$= 12.97\ in^4$

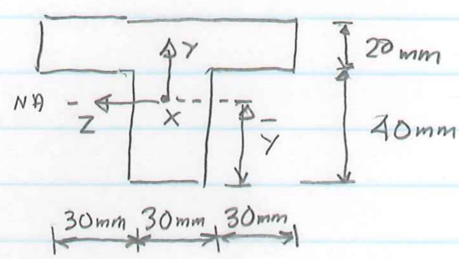
Find M_{allow} for $\sigma_{allow} = 20$ ksi

$M_{allow} = \sigma_{allow} \cdot I/c = 103.8$ kip-in

Find ρ when $M = M_{allow}$

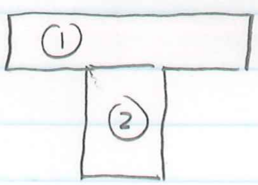
$\rho = EI/M = 110.4$ ft

Example:



Find $\bar{Y}$ (location of neutral axis) I, c, S

distance to centroid of each piece



piece	A_i	$\bar{Y}_i$	$A_i\bar{Y}_i$
1	$1800\ mm^2$	50 mm	$90{,}000\ mm^3$
2	$1200\ mm^2$	20 mm	$24{,}000\ mm^3$

$$\bar{Y} = \frac{\Sigma A_i \bar{Y}_i}{\Sigma A_i} = \frac{114{,}000\ mm^3}{3{,}000\ mm^2} = 38\ mm$$

$c = \max(38\,mm, 60\,mm - 38\,mm) = 38\ mm$

all units mm

$$I = \frac{1}{12}90 \cdot 20^3 + 90 \cdot 20 \cdot (50-38)^2 + \frac{1}{12}30 \cdot 40^3 + 30 \cdot 40 \cdot (20-38)^2$$

$= 868{,}000\ mm^4$

$S = I/c = 22{,}842\ mm^3$ ($\sigma_{max} = M/S$ will give maximum stress at bottom fiber.)